

On the structure of virtually nilpotent compact p -adic analytic groups

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Abstract

Let G be a compact p -adic analytic group. We recall the well-understood finite radical Δ^+ and FC-centre Δ , and introduce a p -adic analogue of Roseblade's subgroup $\text{nio}(G)$, the unique largest orbitally sound open normal subgroup of G . Further, when G is nilpotent-by-finite, we introduce the finite-by-(nilpotent p -valuable) radical $\mathbf{FN}_p(G)$, an open characteristic subgroup of G contained in $\text{nio}(G)$. By relating the already well-known theory of isolators with Lazard's notion of p -saturation, we introduce the isolated lower central (resp. isolated derived) series of a nilpotent (resp. soluble) p -valuable group of finite rank, and use this to study the conjugation action of $\text{nio}(G)$ on $\mathbf{FN}_p(G)$. We emerge with a structure theorem for G ,

$$1 \leq \Delta^+ \leq \Delta \leq \mathbf{FN}_p(G) \leq \text{nio}(G) \leq G,$$

in which the various quotients of this series of groups are well understood. This sheds light on the ideal structure of the Iwasawa algebras (i.e. the completed group rings kG) of such groups, and will be used in future work to study the prime ideals of these rings.

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Introduction

We aim to study the structure of certain compact p -adic analytic groups G . This will crucially underpin later work in which we will explore the ring-theoretic properties of completed group rings kG , where k is a finite field of characteristic p and G is a nilpotent-by-finite compact p -adic analytic group.

There is much in common between the theory of polycyclic-by-finite groups (and their group rings) and the theory of polycyclic-by-finite compact p -adic analytic groups (and their completed group rings). See [4] and [5], or the more recent survey paper by Ardakov and Brown [3], for an overview of the latter.

It is known that

$$\left\{ \begin{array}{c} \text{uniform groups} \\ [4, \text{Definition 4.1}] \end{array} \right\} \subseteq \left\{ \begin{array}{c} p\text{-valuable groups} \\ [5, \text{III, 2.1.2}] \end{array} \right\} \subseteq \left\{ \begin{array}{c} \text{compact } p\text{-adic} \\ \text{analytic groups} \\ [4, \text{Definition 8.14}] \end{array} \right\},$$

with the first inclusion coming from [4, Definition 1.15; notes at end of chapter 4], and the second from [4, Corollary 8.34].

Note also that compact p -adic analytic groups G are profinite groups satisfying **Max**: every nonempty set of closed subgroups of G contains a maximal element. Indeed, in the case when G is p -valuable, this follows from [5, III, 3.1.7.5]; in the general case, G contains a uniform open normal subgroup U by [4, 8.34]. Indeed, [4, 8.34] implies that compact p -adic analytic groups are precisely extensions of uniform (or p -valuable) groups by finite groups.

We aim eventually to extend some of the work of Roseblade [13] and Letzter and Lorenz [6] to the domain of compact p -adic analytic groups, building on work by Ardakov [2]. Our main results are as follows.

Let G be a compact p -adic analytic group, and H a closed subgroup. Following Roseblade [13], we will say that H is *orbital* (or *G -orbital*) if it only has finitely many G -conjugates, or equivalently if its normaliser $\mathbf{N}_G(H)$ is open in G ; and H is *isolated orbital* (or *G -isolated orbital*) if H is orbital, and given any other closed orbital subgroup H' of G with $H \leq H'$, we have $[H' : H] = \infty$. G is then said to be *orbitally sound* if all its isolated orbital closed subgroups are in fact normal.

We define the Roseblade subgroup

$$\text{nio}(G) = \bigcap_H \mathbf{N}_G(H),$$

where this intersection is taken over all isolated orbital closed subgroups H of G . In section 2, we prove:

Theorem A. Let G be a compact p -adic analytic group. Then $\text{nio}(G)$ is an orbitally sound, open, characteristic subgroup of G , and contains all finite-by-nilpotent closed normal subgroups of G . \square

Now let G be a p -valuable group. Ardakov shows, in [2, Lemma 8.4(a)], that the centre $Z(G)$ is isolated orbital in G ; we may then deduce from Lemma 1.5(i) that the usual upper central series

$$1 \leq Z(G) \leq Z_2(G) \leq \dots$$

of G as an abstract group consists of *isolated orbital* subgroups of G , so that any p -valuation on G naturally induces p -valuations on each $Z_i(G)$ and $G/Z_i(G)$. Unfortunately, in general, the (abstract) lower central series $\{\gamma_i\}$ (or derived series $\{\mathcal{D}_i\}$) of G will not necessarily consist of *isolated* orbital subgroups, so G/γ_i (or G/\mathcal{D}_i) will not necessarily remain p -valuable.

In section 3, we introduce appropriate “isolated” analogues of these series for p -valuable groups:

Theorem B. Let G be a p -valuable group. Then there exists a unique fastest descending series of isolated orbital closed normal subgroups of G , the *isolated lower central series*,

$$G = G_1 \triangleright G_2 \triangleright \dots,$$

with the properties that each G_i is characteristic in G , G_i/G_{i+1} is abelian for each i , $[G_i, G_j] \leq G_{i+j}$ for all i and j , and there exists some r with $G_r = 1$ if and only if G is nilpotent.

There exists also a unique fastest descending series of isolated orbital closed normal subgroups of G , the *isolated derived series*,

$$G = G^{(0)} \triangleright G^{(1)} \triangleright \dots,$$

with the properties that each $G^{(i)}$ is characteristic in G , $G^{(i)}/G^{(i+1)}$ is abelian for each i , and there exists some r with $G^{(r)} = 1$ if and only if G is soluble. \square

Now let G be a fixed compact p -adic analytic group. Define the two closed subgroups

$$\begin{aligned} \Delta &= \{x \in G \mid [G : \mathbf{C}_G(x)] < \infty\}, \\ \Delta^+ &= \{x \in \Delta \mid o(x) < \infty\}, \end{aligned}$$

where $o(x)$ denotes the order of x . We will show that we always have an inclusion of subgroups

$$1 \leq \Delta^+ \leq \Delta \leq \text{nio}(G) \leq G.$$

Suppose further that G is a *nilpotent-by-finite* compact p -adic analytic group. Consider the set of finite-by-(nilpotent p -valuable) open normal subgroups H of G – that is, the set of open normal subgroups H that contain a finite normal subgroup $F \triangleleft H$, such that H/F is nilpotent and p -valuable. The main result of section 5 is:

Theorem C. This set contains a unique maximal element H , which is characteristic in G . There is an inclusion of subgroups

$$\Delta \leq H \leq \text{nio}(G),$$

and the quotient group $\text{nio}(G)/H$ is isomorphic to a subgroup of the group of torsion units of \mathbb{Z}_p . \square

We will denote this H by $\mathbf{FN}_p(G)$, the *finite-by-(nilpotent p -valuable) radical* of G .

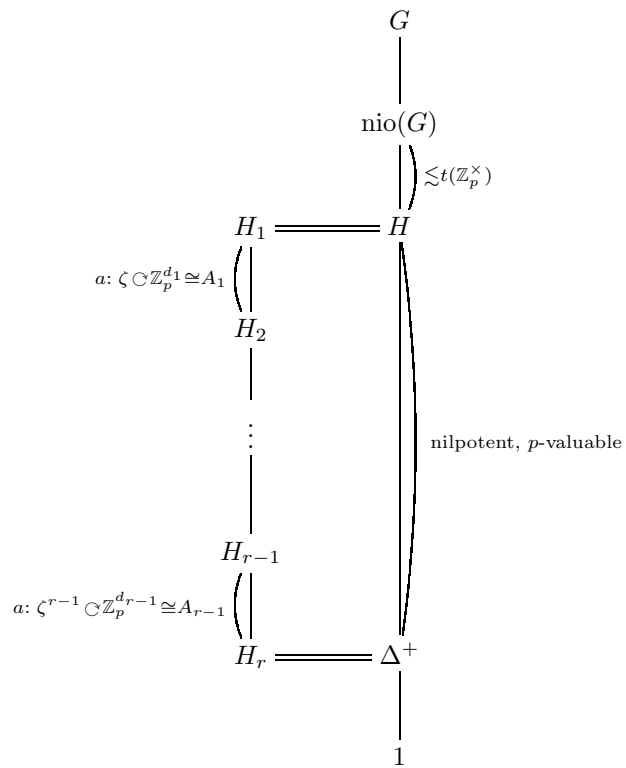
Putting this together with the technical results of section 4, we can now understand the conjugation action of $\text{nio}(G)$ on $\mathbf{FN}_p(G)$:

Theorem D. Let G be a nilpotent-by-finite compact p -adic analytic group, and $H = \mathbf{FN}_p(G)$ its finite-by-(nilpotent p -valuable) radical. Write

$$N = N_1 \triangleright N_2 \triangleright \cdots \triangleright N_r = 1$$

for the isolated lower central series of $N = H/\Delta^+$, and let H_i be the full preimage in G of N_i . Then $\text{nio}(G)/H$ is isomorphic to a subgroup of $t(\mathbb{Z}_p^\times)$, the group of torsion units of \mathbb{Z}_p , and so is cyclic; let a be a preimage in $\text{nio}(G)$ of a generator of $\text{nio}(G)/H$. Then conjugation by a acts on each H_i , and hence induces an action on the (free, finite-rank) \mathbb{Z}_p -modules $H_i/H_{i+1} = A_i$. In multiplicative notation, there is some scalar $\zeta \in t(\mathbb{Z}_p^\times)$ such that $x^a = x^{\zeta^i}$ for all $x \in A_i$. \square

Putting these ingredients together allows us to understand the structure of a nilpotent-by-finite compact p -adic analytic group G , which, for convenience, we display in the following diagram.



1 Preliminaries

Definition 1.1. Let G be a profinite group. A closed subgroup H of G is *G-orbital* (or just *orbital*, when the group G is clear from context) if H has only finitely many G -conjugates, i.e. if $\mathbf{N}_G(H)$ is open in G . Similarly, an element $x \in G$ is *orbital* if $[G : \mathbf{C}_G(x)] < \infty$.

Remark. Note that, if G is compact p -adic analytic, it is profinite (by [4, 8.34]).

Definition 1.2. The *FC-centre* $\Delta(G)$ of an arbitrary group G is the subgroup of all orbital elements of G . The *finite radical* $\Delta^+(G)$ of G is the subgroup of all torsion orbital elements of G .

Remark. Throughout this paper, we will write as shorthand $\Delta^+ = \Delta^+(G)$ and $\Delta = \Delta(G)$. Also throughout this paper, all subgroups will be closed, all homomorphisms continuous, etc. unless otherwise specified.

Lemma 1.3. Let G be a compact p -adic analytic group. For convenience, we record a few basic properties of Δ and Δ^+ .

- (i) Δ^+ is finite.
- (ii) If H is an open subgroup of G , then $\Delta^+(H) \leq \Delta^+(G)$ and $\Delta(H) \leq \Delta(G)$.
- (iii) When G is compact p -adic analytic, Δ^+ and Δ are closed in G .
- (iv) Δ^+ and Δ are characteristic subgroups of G .
- (v) Δ/Δ^+ is a torsion-free abelian group.

Proof.

- (i) Δ^+ is generated by the finite normal subgroups of G [11, 5.1(iii)]. It is obvious that the compositum of two finite normal subgroups is again finite and normal. Now suppose that Δ^+ is infinite, and take an open uniform subgroup H of G [4, 4.3]: then $\Delta^+ \cap H$ is non-trivial, and so we must have some finite normal subgroup F with $F \cap H$ non-trivial. But F is torsion, so this contradicts the fact that H is torsion-free [4, 4.5].
- (ii) If an element $x \in H$ has finitely many H -conjugates, and H has finite index in G , then x has finitely many G -conjugates.
- (iii) Δ^+ is closed because it is finite.

For the case of Δ , suppose first that G is p -valued [5, III, 2.1.2]. Now, any orbital $x \in G$ has $\mathbf{C}_G(x)$ open in G , and so, for any $g \in G$, there exists some n with $g^{p^n} \in \mathbf{C}_G(x)$, i.e. $g^{p^n}x = xg^{p^n}$. This implies that $(g^x)^{p^n} = g^{p^n}$, and so by [5, III, 2.1.4], we get $g^x = g$. Hence $\mathbf{C}_G(x) = G$. In other words, $\Delta = Z(G)$, which is closed in G .

When G is not p -valued, it still has an open p -valued subgroup N [4, 4.3]. Clearly $\Delta(N) = \Delta(G) \cap N$, and so $[\Delta(G) : \Delta(N)] \leq [G : N] < \infty$. So

$\Delta(G)$ is a finite union of translates of $Z(N)$, which is closed in N and hence closed in G .

(iv) See [10, discussion after lemma 4.1.2 and lemma 4.1.6].

(v) See [10, lemma 4.1.6]. \square

Throughout the remainder of this subsection, G is a profinite group unless stated otherwise.

Definition 1.4. An orbital closed subgroup H of G is *isolated* if, for all orbital closed subgroups H' of G with $H \leq H' \leq G$, we have $[H' : H] = \infty$. (We will sometimes say that a closed subgroup is *G -isolated orbital* as shorthand for *isolated as an orbital closed subgroup of G* .) Following Passman [11, definition 19.1], if all isolated orbital closed subgroups of G are in fact normal, we shall say that G is *orbitally sound*.

We record a few basic properties, before showing that this definition is the same as the one given in [13, 1.3] and [2, 5.8] (in Lemma 1.10 below).

Lemma 1.5. Let N be a closed normal subgroup of G .

- (i) Suppose H is a closed subgroup of G containing N . Then H/N is (G/N) -orbital if and only if H is G -orbital; and H/N is (G/N) -isolated orbital if and only if H is G -isolated orbital.
- (ii) Suppose G is orbitally sound. Then G/N is orbitally sound.
- (iii) Suppose N is finite and G/N is orbitally sound. Then G is orbitally sound.

Proof.

- (i) It is easily checked that $\mathbf{N}_{G/N}(H/N) = \mathbf{N}_G(H)/N$, and so

$$[G : \mathbf{N}_G(H)] = [G/N : \mathbf{N}_G(H)/N] = [G/N : \mathbf{N}_{G/N}(H/N)].$$

So H is orbital if and only if H/N is orbital. Suppose these two groups are both orbital, and let H' be an orbital closed subgroup of G with $H \leq H' \leq G$; then $[H' : H] = [H'/N : H/N]$, so H is isolated if and only if H' is isolated.

- (ii) Let H/N be an isolated orbital closed subgroup of G/N . Then, by (i), H is an isolated orbital subgroup of G , so $H \triangleleft G$, and so $H/N \triangleleft G/N$.
- (iii) Let H be an isolated orbital closed subgroup of G , and H' an orbital closed subgroup of G with $H \leq H' \leq G$. If H contains N , then we may apply (i) to show that $[H' : H] = [H'/N : H/N] = \infty$ as G/N is orbitally sound. But H must contain N : indeed, as H is G -orbital, clearly HN/N is G/N -orbital, and so, by (i), HN is G -orbital. But N is finite, so $[HN : H] < \infty$, and H is isolated, so $H = HN$. \square

From now on, we assume that G is a profinite group satisfying the *maximum condition on closed subgroups*: every nonempty set of closed subgroups of G has a maximal element.

Remark. Note that, if G is compact p -adic analytic, it satisfies the maximum condition on closed subgroups. Indeed, this is true for p -valuable G by [5, III, 3.1.7.5], and hence true for any compact p -adic analytic group G , as G contains a uniform (hence p -valuable) subgroup of finite index [4, 8.34].

Definition 1.6. If H is an orbital closed subgroup of G , we define its *isolator* $i_G(H)$ in G to be the closed subgroup of G generated by all orbital closed subgroups L of G containing H as an open subgroup, i.e. with $[L : H] < \infty$.

Once we have proved that $i_G(H)$ is indeed an isolated orbital closed subgroup of G containing H as an open subgroup, it will be clear from the definition that it is the unique such closed subgroup.

We now prove some basic properties of $i_G(H)$, following [11].

Proposition 1.7. Suppose H is an orbital closed subgroup of G . Then H is open in $i_G(H)$.

Proof. We first show that, if L_1 and L_2 are orbital subgroups of G containing H as an open subgroup, then $[\langle L_1, L_2 \rangle : H] < \infty$. Write $(-)^{\circ}$ for $\bigcap_{g \in G} (-)^g$. Suppose without loss of generality that $G = \langle L_1, L_2 \rangle$, and that $H^{\circ} = 1$ (by passing to G/H°).

For $i = 1, 2$, as $[L_i : H] < \infty$ and as H, L_i are all orbital, we may take an open normal subgroup N of G such that $[N, L_i] \subseteq H$. Indeed, $\mathbf{N}_G(L_i)$ is a subgroup of finite index in G , and permutes the (finitely many) left cosets of H in L_i by left multiplication; take N_i to be the kernel of this action, and set $N = N_1 \cap N_2$.

Hence $[N \cap H, L_i] \subseteq N \cap H$, i.e. both L_1 and L_2 normalise $N \cap H$, so G normalises $N \cap H$. So $N \cap H$ is a normal subgroup of G contained in H , and by assumption must be trivial. But N was an open subgroup of G , so H must have been finite, and so L_1 and L_2 must be *finite* orbital subgroups of G . This implies that $L_i \leq \Delta^+$, and hence $G = \Delta^+$, so that G is finite, as required.

Now, in the general case, G satisfies the maximal condition on closed subgroups, so we can choose L maximal subject to L being orbital and $[L : H] < \infty$. This L is $i_G(H)$ and contains H as an open subgroup. \square

Lemma 1.8.

- (i) Suppose H is an orbital closed subgroup of G . Then $i_G(H)$ is an isolated orbital closed subgroup of G . Furthermore, if H is normal in G , then so is $i_G(H)$.
- (ii) Suppose G is orbitally sound and H is a closed subgroup of finite index. Then H is orbitally sound.

Proof.

- (i) If $i_G(H)$ is orbital, then by Proposition 1.7, it is isolated (by construction). But H has finite index in $i_G(H)$, so $i_G(H)$ must be generated by a finite number of subgroups L_1, \dots, L_n containing H as a subgroup of finite index. So

$$\bigcap_{i=1}^n \mathbf{N}_G(L_i) \leq \mathbf{N}_G(i_G(H)),$$

and as each $\mathbf{N}_G(L_i)$ is open in G , so is $\mathbf{N}_G(i_G(H))$.

Now suppose that H is normal in G . To see that $i_G(H)$ is normal in G , fix $g \in G$, and note that conjugation by g fixes H and therefore simply permutes the set of orbital closed subgroups L of G containing H as an open subgroup, i.e. permutes the set of subgroups of G that generate $i_G(H)$ (see Definition 1.6).

- (ii) Let K be an isolated H -orbital closed subgroup of H . Then K is G -orbital, so $i_G(K)$ is an isolated orbital subgroup of G , and so is normal in G . Hence $i_G(K) \cap H$ is normal in H . But $[i_G(K) : K] < \infty$, so $[i_G(K) \cap H : K] < \infty$, and hence $i_G(K) \cap H = K$, as K was assumed to be isolated in H . \square

Lemma 1.9. Let H be an open normal subgroup of G . Then there is a one-to-one correspondence

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{isolated orbital} \\ \text{closed subgroups of } G \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{isolated orbital} \\ \text{closed subgroups of } H \end{array} \right\} \\ G' \longmapsto & & i_H(G' \cap H), \\ i_G(H') \longleftarrow & & H'. \end{array}$$

Proof. Suppose first that H' is an arbitrary orbital closed subgroup of H . That is, $\mathbf{N}_H(H')$ is open in H , hence also in G , and so $\mathbf{N}_G(H')$ must be open in G . Therefore H' is also G -orbital, and so, by Lemma 1.8(i), $i_G(H')$ is an isolated orbital closed subgroup of G .

Conversely, take a G -isolated orbital closed subgroup G' : then $\mathbf{N}_G(G')$ is open in G , as G' is G -orbital, which implies that $\mathbf{N}_H(G' \cap H)$ is open in H , i.e. that $G' \cap H$ is H -orbital. Now $i_H(G' \cap H)$ is H -isolated orbital by Lemma 1.8(i).

Now we will show that these correspondences are mutually inverse.

In one direction, we must take H' as above, assume further that it is H -isolated orbital, and show that $i_H(i_G(H') \cap H) = H'$. To do this, note that $i_G(H')$ contains H' as an open subgroup by Proposition 1.7, so $i_G(H') \cap H$ is an H -orbital closed subgroup (by the correspondence above) containing the H -isolated orbital H' as an open subgroup, and so by definition the two must be equal.

For the converse direction, we must take G' as above and show that $i_G(i_H(G' \cap H))$ is equal to G' . But clearly $i_G i_H = i_G$, and both G' and $i_G(G' \cap H)$ are G -isolated

orbital (G' by assumption, $i_G(G' \cap H)$ by definition) and contain $G' \cap H$ as an open subgroup, so by uniqueness (see Definition 1.6), they are equal. \square

Lemma 1.10. The following are equivalent:

- (i) Any isolated orbital closed subgroup H of G is normal.
- (ii) Any orbital closed subgroup K of G contains a subgroup N of finite index in K which is normal in G .

Proof.

(i) \Rightarrow (ii) Let K be an orbital closed subgroup of G . By Lemma 1.8(i), $i_G(K)$ is an isolated orbital closed subgroup of G , and so (by assumption) is normal in G . Therefore, as it contains K as a subgroup of finite index (by Proposition 1.7), it must contain each conjugate K^g (for any $g \in G$) as a subgroup of finite index. But as K is G -orbital, it only has finitely many G -conjugates, and so their intersection K° still has finite index in $i_G(K)$ and hence also in K , and K° is normal in G by construction.

(ii) \Rightarrow (i) Let H be an isolated orbital closed subgroup of G , and write H° for the largest normal subgroup of G contained in H , which by (ii) must have finite index in H . Now clearly $H \leq i_G(H^\circ)$ by definition of $i_G(H^\circ)$, but also $i_G(H^\circ) \leq H$ as H is isolated and contains H° . So H is the G -isolator of a normal subgroup, and so by Lemma 1.8(i), H is also normal in G . \square

2 The Roseblade subgroup $\text{nio}(G)$

We begin this section by remarking that “orbitally sound” is not too restrictive a condition. Recall:

Lemma 2.1. Let G be a p -valuable group. Then G is orbitally sound. \square

Proof. This is [2, Proposition 5.9], after remarking that the definitions of “orbitally sound” given in Definition 1.4 and in [2, 5.8] are equivalent by Lemma 1.10. \square

The following two lemmas will allow us to find a large class of orbitally sound groups.

For the next lemma, fix the following notation. Let G be a compact p -adic analytic group, and consider its \mathbb{Q}_p -Iwasawa algebra $\mathbb{Q}_p G := (\mathbb{Z}_p G) \left[\frac{1}{p} \right]$. Write I for its augmentation ideal

$$I = \ker(\mathbb{Q}_p G \rightarrow \mathbb{Q}_p).$$

Recall that I^k is generated over $\mathbb{Q}_p G$ by $\{(x_1 - 1) \dots (x_k - 1) \mid x_i \in G\}$. Now it is clear that G acts unipotently on the series

$$\mathbb{Q}_p G > I > I^2 > I^3 > \dots,$$

i.e. for all $g \in G$, we have $(g - 1)\mathbb{Q}_p G \subseteq I$ and $(g - 1)I^k \subseteq I^{k+1}$.

Write also \mathcal{U}_n for the subgroup of $GL_n(\mathbb{Q}_p)$ consisting of upper triangular unipotent matrices.

Lemma 2.2. Write

$$D_k = \ker(G \rightarrow (\mathbb{Q}_p G / I^k)^\times),$$

the k -th rational dimension subgroup of G , for all $k \geq 1$. Then the D_k are a descending chain of isolated orbital closed normal subgroups of G . This chain eventually stabilises: that is, there exists some t such that $D_n = D_t$ for all $n \geq t$.

Furthermore, if G is torsion-free and nilpotent, $D_t = 1$, and G is isomorphic to a closed subgroup of \mathcal{U}_m for some m .

Proof. By definition, it is clear that the D_k are closed normal (hence orbital) subgroups of G ; to show that they are isolated orbital, we will show that each G/D_k is torsion-free.

Fix k . Consider the series of finite-dimensional \mathbb{Q}_p -vector spaces

$$\mathbb{Q}_p G / I^k > I / I^k > I^2 / I^k > I^3 / I^k > \dots > I^k / I^k,$$

and choose a basis for $\mathbb{Q}_p G/I^k$ which is filtered relative to this series: i.e. by repeatedly extending a basis for I^r/I^k to a basis for I^{r-1}/I^k , we get a basis

$$B = \{e_1, \dots, e_r\}$$

and integers

$$0 = n_k < n_{k-1} < n_{k-2} < \dots < n_1 < n_0 = r$$

with the property that $\{e_1, \dots, e_{n_r}\}$ is a basis for I^r/I^k for each $0 \leq r \leq k$ (where we write $I^0 = \mathbb{Q}_p G$ for convenience).

As G acts *unipotently* (by left multiplication) on $\mathbb{Q}_p G/I^k$, and by definition of the basis B , we see that with respect to B , each $g \in G$ acts by a unipotent upper-triangular matrix, i.e. we get a continuous group homomorphism $G \rightarrow \mathcal{U}_r$. Now D_k is just the kernel of this map; but \mathcal{U}_r is torsion-free, so D_k must be isolated.

Recall the *dimension* $\dim H$ of a pro- p group H of finite rank from [4, 4.7]. As G has finite rank, it also has finite dimension [4, 3.11, 3.12], and we must have $\dim D_i \geq \dim D_{i+1}$ for all i by [4, 4.8]. But if $\dim D_i = \dim D_{i+1}$, then D_i/D_{i+1} is a p -valued group (as D_{i+1} is isolated) of dimension 0 (again by [4, 4.8]), and so must be trivial. Hence the sequence (D_i) stabilises after at most $t := 1 + \dim G$ terms, and so

$$D_t = \bigcap_{n \geq 1} D_n.$$

Now suppose that G is nilpotent. Then, by [1, Theorem A], it follows that I is localisable. Let $R = (\mathbb{Q}_p G)_I$ be its localisation, and $J(R) = \mathfrak{m}$ its unique maximal ideal: then the ideal

$$A = \bigcap_{n \geq 1} \mathfrak{m}^n$$

satisfies $A = \mathfrak{m}A$, so by Nakayama's lemma [7, 0.3.10], we must have $A = 0$. This implies that

$$\bigcap_{n \geq 1} I^n \subseteq \ker(\mathbb{Q}_p G \rightarrow R).$$

Assuming further that G is torsion-free, we see that $\mathbb{Q}_p G$ is a domain [9, Theorem 1], and so the localisation map $\mathbb{Q}_p G \rightarrow R$ is injective. Hence $\bigcap_{n \geq 1} I^n = 0$, and so

$$D_t = \bigcap_{n \geq 1} D_n = \left(\bigcap_{n \geq 1} (I^n + 1) \cap G \right) \subseteq \left(\bigcap_{n \geq 1} I^n \right) + 1 = 1.$$

Now the representation $G \rightarrow \text{Aut}(\mathbb{Q}_p G/I^t) \cong GL_m(\mathbb{Q}_p)$ is faithful and has image in \mathcal{U}_m . \square

Lemma 2.3. Let G be a (topologically) finitely generated nilpotent pro- p group. Then G is p -valuable if and only if it is torsion-free.

Proof. Lemma 2.2 gives an injective map $G \rightarrow \mathcal{U}_m$. Now, as G is topologically finitely generated, its image in \mathcal{U}_m must lie inside the set $\frac{1}{p^t} M_m(\mathbb{Z}_p) \cap \mathcal{U}_m$ for some t . Hence, by conjugating by the diagonal element

$$\text{diag}(p, p^2, \dots, p^m)^{t+\varepsilon} \in GL_m(\mathbb{Q}_p),$$

where

$$\varepsilon = \begin{cases} 1 & p > 2, \\ 2 & p = 2, \end{cases}$$

we see that G is isomorphic to a subgroup of

$$\Gamma_\varepsilon = \{ \gamma \in GL_m(\mathbb{Z}_p) \mid \gamma \equiv 1 \pmod{p^\varepsilon} \},$$

the ε th congruence subgroup of $GL_m(\mathbb{Z}_p)$, which is uniform (and hence p -valuable) by [4, Theorem 5.2].

The reverse implication is clear from the definition of a p -valuation [5, III, 2.1.2]. \square

Now we have found a large class of orbitally sound compact p -adic analytic groups.

Corollary 2.4. If G is a finite-by-nilpotent compact p -adic analytic group, then it is orbitally sound.

Proof. $\overline{G} := G/\Delta^+$ must be a nilpotent compact p -adic analytic group with $\Delta^+(\overline{G}) = 1$, and so \overline{G} is torsion-free by [12, 5.2.7]. Now Lemma 2.3 shows that \overline{G} is p -valuable, and from Lemma 2.1 we may deduce that \overline{G} is orbitally sound. But now Lemma 1.5(iii) implies that G is orbitally sound. \square

Remark. It is well known that finite-by-nilpotent implies nilpotent-by-finite, but not conversely. Not all nilpotent-by-finite compact p -adic analytic groups are orbitally sound: indeed, the wreath product

$$G = \mathbb{Z}_p \wr C_2 = (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes C_2$$

is *abelian*-by-finite, and the infinite procyclic subgroup $H = \mathbb{Z}_p \times \{0\}$ is orbital, but the largest G -normal subgroup contained in H is the trivial subgroup.

We can now define the Roseblade subgroup.

Definition 2.5. As in Roseblade [13, 1.3], write $\text{nio}(G)$ for the closed subgroup of G defined by

$$\text{nio}(G) = \bigcap_H \mathbf{N}_G(H),$$

where the intersection ranges over the isolated orbital closed subgroups H of G .

Theorem 2.6. Let G be a compact p -adic analytic group.

- (i) An orbitally sound open normal subgroup $N \triangleleft G$ normalises every closed isolated orbital subgroup $H \leq G$.
- (ii) $\text{nio}(G)$ is an orbitally sound open characteristic subgroup of G .

Proof.

- (i) Since H is isolated orbital in G , we must have that $H \cap N$ is isolated orbital (and hence normal) in N . However, it follows from Proposition 1.7 that $H = \text{i}_G(H \cap N)$. Hence H is generated by all the (finitely many) closed orbital subgroups L_1, \dots, L_k of G containing $H \cap N$ as a subgroup of finite index. Conjugation by $n \in N$ permutes these L_i , and so fixes H .
- (ii) Let N be a complete p -valued open normal subgroup of G (e.g. by [4, 8.34]). Then [2, Proposition 5.9] shows that N is orbitally sound, and hence by (i) N normalises all closed isolated orbital subgroups of G . So, by definition, $N \leq \text{nio}(G)$, and so $[G : \text{nio}(G)] \leq [G : N] < \infty$. Therefore $\text{nio}(G)$ is open in G as required. But by definition, $\text{nio}(G)$ is the largest subgroup that normalises all isolated orbital subgroups of G , so by the correspondence of Lemma 1.9 and Lemma 1.8(i), it normalises all isolated orbital subgroups of $\text{nio}(G)$, i.e. it is orbitally sound. \square

Proof of Theorem A. It is clear from Definition 2.5 and Theorem 2.6(i), (ii) that $\text{nio}(G)$ is the unique maximal *orbitally sound* closed normal subgroup of G , and is hence characteristic in G . Corollary 2.4, together with Fitting's theorem [14, 1B, Proposition 15], implies that it contains all finite-by-nilpotent closed normal subgroups. \square

3 p -saturations

Recall [5, III, 2.1.2] that a p -valuation of a group G is a function

$$\omega : G \rightarrow \mathbb{R} \cup \{\infty\}$$

satisfying the following properties:

- $\omega(x) = \infty$ if and only if $x = 1$,
- $\omega(x) > (p-1)^{-1}$,
- $\omega(x^{-1}y) \geq \inf\{\omega(x), \omega(y)\}$,
- $\omega([x, y]) \geq \omega(x) + \omega(y)$,
- $\omega(x^p) = \omega(x) + 1$,

for all $x, y \in G$. The group G , when endowed with the p -valuation ω , is called p -valued. On the other hand, a group G is called p -valuable [5, III, 3.1.6] if there exists a p -valuation ω of G with respect to which G is complete of finite rank.

Let G be a p -valuable group, and fix a p -valuation ω on G , so that G is complete p -valued of finite rank. Recall the definition of the p -saturation $\text{Sat } G$ of G (with respect to ω) from [5, IV, 3.3.1.1]: this is again a complete p -valued group of finite rank, and there is a natural isometry identifying G with an open subgroup of $\text{Sat } G$ [5, IV, 3.3.2.1]. We will prove a few basic facts about p -saturations.

Firstly, we will prove a basic relationship between isolators and p -saturations.

Lemma 3.1. Let G be a complete p -valued group of finite rank, and let H be a closed normal (and hence orbital) subgroup of G . Then $i_G(H) = \text{Sat } H \cap G$ (considered as subgroups of $\text{Sat } G$).

Proof. $[\text{Sat } H : H] < \infty$ by [5, IV, 3.4.1], and $\text{Sat } H$ is a closed normal subgroup of $\text{Sat } G$ by [5, IV, 3.3.3], so $S := \text{Sat } H \cap G$ is a normal (and hence orbital) subgroup of G , and contains H as a subgroup of finite index. Hence, by Definition 1.6, S is contained in $i_G(H)$.

To show the reverse inclusion, we will consider the group $i_G(H)/S$, which is a finite subgroup of G/S (as it is a quotient of $i_G(H)/H$, which is finite by Proposition 1.7). But G/S is isomorphic to $G\text{Sat } H/\text{Sat } H$, a subgroup of the torsion-free group $\text{Sat } G/\text{Sat } H$ (see [5, IV, 3.4.2] or [5, III, 3.3.2.4]). In particular, G/S has no non-trivial finite subgroups, so we must have $i_G(H) = S$. \square

Remark. Of course, $i_G(H)$ is independent of the choice of ω .

Lemma 3.2. Let G be a complete p -valued group of finite rank, which we again identify with an open subgroup of its p -saturation S . Suppose S' is a p -saturated closed normal subgroup of S , and set $G' = S' \cap G$. Then there is a natural isometry $S/S' \cong \text{Sat } (G/G')$.

Proof. We will show that S/S' satisfies the universal property for $\text{Sat}(G/G')$ [5, IV, 3.3.2.4]. Clearly we may regard $G/G' \cong GS'/S'$ as a subgroup of S/S' . Note also that S/S' is p -saturated, by [5, III, 3.3.2.4]. Also, as G' is open in S' and S' is p -saturated, we have that $S' = \text{Sat } G'$.

Let H be an arbitrary p -saturated group and $\varphi : G/G' \rightarrow H$ a homomorphism of p -valued groups. We must first construct a map $\psi : S/S' \rightarrow H$. To do this, we first compose φ with the natural surjection $G \rightarrow G/G'$ to get a map $\alpha : G \rightarrow H$, which we may then extend uniquely to a map $\beta : S \rightarrow H$ using the universal property of $S = \text{Sat } G$, so that $\alpha = \beta|_G$ and the following diagram commutes.

$$\begin{array}{ccccc} & & \alpha & & \\ & \nearrow & & \searrow & \\ G & \twoheadrightarrow & G/G' & \xrightarrow{\varphi} & H \\ \downarrow & & & \nearrow \beta & \\ S & & & & \end{array}$$

Now we wish to show that β descends to a map $S/S' \rightarrow H$. To do this, we first study the restriction of α to G' and of β to S' . The following diagram commutes:

$$\begin{array}{ccc} G' & \xrightarrow{\alpha|_{G'}} & H \\ \downarrow & \nearrow \beta|_{S'} & \\ S' & & \end{array}$$

and so, since $S' = \text{Sat } G'$, $\beta|_{S'}$ must be the *unique* extension of $\alpha|_{G'}$ to a map $S' \rightarrow H$, as $S' = \text{Sat } G'$. But α factors through G/G' , i.e. $\alpha|_{G'}$ is the trivial homomorphism $G' \rightarrow H$, so it extends to the trivial homomorphism $S' \rightarrow H$. By uniqueness, we must have $S' \subseteq \ker \beta$. This shows that β induces a map $\psi : S/S' \rightarrow H$.

Finally, suppose $\varphi : G/G' \rightarrow H$ has two distinct extensions $\psi_1, \psi_2 : S/S' \rightarrow H$. Then we may compose them with the natural surjection $S \rightarrow S/S'$ to get two distinct maps $\beta_1, \beta_2 : S \rightarrow H$. Their restrictions $\alpha_1, \alpha_2 : G \rightarrow H$ to G must therefore also be distinct, for if not, then the map $\alpha_1 = \alpha_2 : G \rightarrow H$ has (at least) two distinct extensions to maps $S \rightarrow H$, contradicting the universal property of $S = \text{Sat } G$. Finally, if α_1 and α_2 are distinct, then they descend to distinct maps $\varphi_1, \varphi_2 : G/G' \rightarrow H$, contradicting our assumption. So the extension of φ to ψ is unique. \square

Remark. Lemma 3.2 holds even if G does not have finite rank, and hence is only closed (not necessarily open) in its p -saturation S .

Definition 3.3. Let G be an arbitrary group. A *central series* for G is a sequence of subgroups

$$G = G_1 \triangleright G_2 \triangleright \cdots \triangleright G_n = 1$$

with the property that $[G, G_i] \leq G_{i+1}$ for each i . (For the purposes of this definition, G_j is understood to mean 1 if $j > n$, and G if $j < 1$.)

We will say that a central series is *strongly* central if also $[G_i, G_j] \leq G_{i+j}$ for all i and j .

An *abelian series* for G a sequence of subgroups

$$G = G_1 \triangleright G_2 \triangleright \cdots \triangleright G_n = 1$$

with the property that $[G_i, G_i] \leq G_{i+1}$ for each i .

When G is a topological group, we will insist further that all of the G_i should be *closed* subgroups of G .

Remark. We will be working with nilpotent p -valuable groups G . It will be useful for us to define the *isolated lower central series* of G , which will turn out to be the fastest descending central series of closed subgroups

$$G = G_1 \triangleright G_2 \triangleright \cdots \triangleright G_r = 1$$

with the property that the successive quotients G_i/G_{i+1} are torsion-free (and hence p -valuable, by [5, IV, 3.4.2]). We will also prove that the isolated lower central series is a *strongly* central series. (We demonstrate an *isolated derived series* for soluble p -valued groups at the same time.)

Lemma 3.4. Let G be a complete p -valued group of finite rank, and $G_1 \geq G_2$ closed normal subgroups of G with G_1/G_2 an abelian pro- p group (which is not necessarily p -valued). Let $S_i = \text{Sat } G_i$ for $i = 1, 2$. Then S_1/S_2 is abelian and torsion-free (and hence p -valued), and has the same rank as G_1/G_2 as a \mathbb{Z}_p -module.

Proof. As S_2 is p -saturated, S_1/S_2 is torsion-free, and so

$$G_1/(S_2 \cap G_1) \cong G_1 S_2 / S_2 \leq S_1 / S_2$$

is torsion-free. G_1/G_2 maps onto $G_1/(S_2 \cap G_1)$ with finite kernel (by Lemma 3.1 and Proposition 1.7, and the assumption that G has finite rank), so $G_1/(S_2 \cap G_1)$ is abelian of the same \mathbb{Z}_p -rank as G_1/G_2 . By Lemma 3.2, S_1/S_2 is the p -saturation of $G_1/(S_2 \cap G_1)$, so is still abelian of the same \mathbb{Z}_p -rank. \square

Before proving the main result of this section, we first need a technical lemma.

Lemma 3.5. Let G be a complete p -valued group of finite rank, and let H and N be two closed normal subgroups. Then

$$[i_G(H), i_G(N)] \leq i_G(\overline{[H, N]}).$$

Proof. Write $L := i_G(\overline{[H, N]})$. This is normal in G , as G is orbitally sound [2, 5.9], and the quotient G/L is still p -valued as it is torsion-free [5, IV, 3.4.2].

Suppose first that $L = 1$, so that $[H, N] = 1$. Then, for any $h \in i_G(H)$, there is some integer n such that $h^{p^n} \in H$, so that $[g, h^{p^n}] = 1$ for all $g \in N$. But this is the same as saying that $h^{p^n} = (h^g)^{p^n}$; and, as G is p -valued, [5, III, 2.1.4] implies that $h = h^g$. As g and h were arbitrary, we see that $[i_G(H), N] = 1$. Repeat this argument for N to show that $[i_G(H), i_G(N)] = 1$.

If $L \neq 1$, we may pass to G/L . Write $\pi : G \rightarrow G/L$ for the natural surjection, so that

$$\pi([i_G(H), i_G(N)]) = [\pi(i_G(H)), \pi(i_G(N))]. \quad (1)$$

Now, $\pi(H)$ is a closed orbital subgroup of $\pi(G)$, and $\pi(i_G(H))$ is a closed orbital subgroup of $\pi(G)$ containing $\pi(H)$ as an open subgroup, so that

$$i_{\pi(G)}(\pi(H)) \geq \pi(i_G(H)),$$

and similarly for N . Together with (1), this implies that

$$\pi([i_G(H), i_G(N)]) \leq [i_{\pi(G)}(\pi(H)), i_{\pi(G)}(\pi(N))].$$

But the right-hand side is now equal to $\pi(1)$, by the previous case, which shows that $[i_G(H), i_G(N)] \leq L$ as required. \square

Corollary 3.6. Let G be a p -valuable group. Define two series:

$$G_i = i_G(\overline{\gamma_i}), \text{ where } \begin{cases} \gamma_1 = G, \\ \gamma_{i+1} = [\gamma_i, G] \end{cases} \text{ for } i \geq 1;$$

$$G^{(i)} = i_G(\overline{\mathcal{D}_i}), \text{ where } \begin{cases} \mathcal{D}_0 = G, \\ \mathcal{D}_{i+1} = [\mathcal{D}_i, \mathcal{D}_i] \end{cases} \text{ for } i \geq 0,$$

where the bars denote topological closure inside G . If G is nilpotent, then (G_i) is a strongly central series for G , i.e. a central series in which $[G_i, G_j] \leq G_{i+j}$. If G is soluble, then $(G^{(i)})$ is an abelian series for G . The quotients G_i/G_{i+1} and $G^{(i)}/G^{(i+1)}$ are torsion-free, and hence p -valuable.

Remark. We prove this using p -saturation, but the resulting closed subgroups G_i and $G^{(i)}$ are independent of the choice of p -valuation ω on G .

The series $(G^{(i)})$ above is a generalisation of the series studied in [8, proof of lemma 2.2.1], there called (G_i) .

Proof. Fix a p -valuation ω on G throughout.

Firstly, we will show that (G_i) is an abelian series. The claim that $(G^{(i)})$ is an abelian series will follow by an identical argument.

The (abstract) lower central series (γ_i) is an abelian series for G as an abstract group (i.e. the subgroups γ_i are not necessarily closed in G), and so the series $(\overline{\gamma_i})$ is a series of *closed* normal subgroups of G , which is still an abelian series by continuity. Now, applying Lemma 3.4 shows that $(\text{Sat } \overline{\gamma_i})$ is also an abelian

series; and by Lemma 3.1, we see that $G_i = \text{Sat } \overline{\gamma_i} \cap G$ for each i , so that (G_i) is an abelian series.

Secondly, we address the claim that the quotients G_i/G_{i+1} are torsion-free and hence p -valuable: this follows from [5, III, 3.1.7.6 / IV, 3.4.2], as the G_{i+1} are isolated in G . The case of the quotients $G^{(i)}/G^{(i+1)}$ is again identical.

Thirdly, we must show that G_{i-1}/G_i is central in G/G_i . Certainly $\gamma_{i-1}G_i/G_i$ is central in G/G_i , because $\gamma_i \leq G_i$, and so

$$\overline{\gamma_{i-1}}G_i/G_i \leq Z(G/G_i)$$

by continuity. However, [2, lemma 8.4(a)] says that $Z(G/G_i)$ is isolated in G/G_i , so by taking (G/G_i) -isolators of both sides, we must have

$$i_{G/G_i}(\overline{\gamma_{i-1}}G_i/G_i) \leq Z(G/G_i);$$

and the left-hand side is clearly equal to G_{i-1}/G_i by Lemma 1.5(i) and Definition 1.6.

Finally, note that

$$[\gamma_i, \gamma_j] \leq \gamma_{i+j}$$

by [12, 5.1.11(i)], and so by taking closures,

$$\overline{[\gamma_i, \gamma_j]} \leq \overline{\gamma_{i+j}}.$$

But $[\overline{\gamma_i}, \overline{\gamma_j}] \leq \overline{[\gamma_i, \gamma_j]}$, as the function $G \times G \rightarrow G$ given by $(a, b) \mapsto [a, b]$ is continuous. Hence

$$[\overline{\gamma_i}, \overline{\gamma_j}] \leq \overline{\gamma_{i+j}},$$

which implies

$$\overline{[\overline{\gamma_i}, \overline{\gamma_j}]} \leq \overline{\gamma_{i+j}},$$

and so, by Lemma 3.5, we may take isolators to show that

$$[i_G(\overline{\gamma_i}), i_G(\overline{\gamma_j})] \leq i_G(\overline{[\overline{\gamma_i}, \overline{\gamma_j}]}) \leq i_G(\overline{\gamma_{i+j}}),$$

i.e. $[G_i, G_j] \leq G_{i+j}$. □

Definition 3.7. When G is a nilpotent (resp. soluble) p -valued group of finite rank, the series (G_i) (resp. $(G^{(i)})$) defined in Corollary 3.6 is the *isolated lower central series* (resp. *isolated derived series*) of G .

Proof of Theorem B. This is the content of Corollary 3.6. □

4 Conjugation action of G

In this subsection, we will study how nilpotent-by-finite compact p -adic analytic groups G act by conjugation on certain torsion-free abelian and nilpotent subquotients. First, we slightly extend the term “orbitally sound”.

Definition 4.1. Let G and H be profinite groups, and suppose G acts (continuously) on H . Then G permutes the closed subgroups of H . We say that the action of G on H is *orbitally sound* if, for any closed subgroup K of H with finite G -orbit, there exists an open subgroup K' of K which is normalised by G .

Recall the group of torsion units of \mathbb{Z}_p :

$$t(\mathbb{Z}_p^\times) = \begin{cases} \{\pm 1\} & p = 2 \\ \mathbb{F}_p^\times & p > 2. \end{cases}$$

Lemma 4.2. Let A be a free abelian pro- p group of finite rank. Let G be a profinite group acting orbitally soundly and by automorphisms of finite order on A . Then, for each $g \in G$, there exists

$$\zeta = \zeta_g \in t(\mathbb{Z}_p^\times)$$

such that $g \cdot x = \zeta x$ for all $x \in A$. This is multiplicative in G , in the sense that $\zeta_g \zeta_h = \zeta_{gh}$ for all $g, h \in G$.

Proof. Write φ for the automorphism of A given by conjugation by g . We may view φ as an automorphism of the \mathbb{Q}_p -vector space $A_{\mathbb{Q}_p} := A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

As the action of G on A is orbitally sound, in particular, we have

$$\langle x \rangle \cap \langle \varphi(x) \rangle \neq \{0\}$$

(as \mathbb{Q}_p -vector subspaces) for every $x \in A_{\mathbb{Q}_p}$. But this just means that x is an eigenvector of the linear map φ . If all elements of $A_{\mathbb{Q}_p}$ are eigenvectors of φ , then they must have a common eigenvalue, say ζ . The statement that G acts on A by automorphisms of finite order means that the eigenvalue ζ for x is of finite order, $\zeta \in t(\mathbb{Z}_p^\times)$.

Multiplicativity is clear from the fact that $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$. \square

Remark. Assume that G is a nilpotent-by-finite, orbitally sound compact p -adic analytic group. In the case when H is an open subgroup of G containing Δ^+ , with the property that $N := H/\Delta^+$ is nilpotent p -valuable, we may consider the isolated lower central series of Corollary 3.6 for N :

$$N = N_1 \triangleright N_2 \triangleright \cdots \triangleright N_r = 1,$$

and take their preimages in G to get a series of characteristic subgroups of H :

$$H = H_1 \triangleright H_2 \triangleright \cdots \triangleright H_r = \Delta^+,$$

with the property that each $A_i := H_i/H_{i+1}$ is a free abelian pro- p group of finite rank.

G clearly acts orbitally soundly on each A_i , as G is itself orbitally sound. Furthermore, as $[H, H_i] \leq H_{i+1}$ for each i , we see that the action $G \rightarrow \text{Aut}(A_i)$ contains the open subgroup H in its kernel, and so G acts by automorphisms of finite order. Thus we may apply Lemma 4.2 to see that G acts on each A_i via a homomorphism $\xi_i : G \rightarrow t(\mathbb{Z}_p^\times)$.

That is, given any $g \in G$ and $h \in H_i$, and writing $\zeta = \xi_i(g)$ and $a = nN_{i+1} \in A_i$, we have

$$(n^g)n^{-\zeta} \in N_{i+1},$$

or equivalently (still in multiplicative notation)

$$a^g = a^\zeta.$$

We now show that the action of an automorphism of G on the quotients A_i is strongly controlled by its action on A_1 . This is an important property that the isolated lower central series shares with the usual lower central series of abstract nilpotent groups; cf. [12, 5.2.5] and the surrounding discussion.

Lemma 4.3. Let H be a finite-by-(nilpotent p -valuable) group, and continue to write $A_i := H_i/H_{i+1}$ as in the remark above. Let α be an automorphism of H inducing multiplication by $\zeta_i \in t(\mathbb{Z}_p^\times)$ on each A_i . Then $\zeta_i = \zeta_1^i$ for each i .

Proof. The map

$$\begin{aligned} A_1 \otimes_{\mathbb{Z}_p} A_i &\rightarrow A_{i+1} \\ xH_2 \otimes yH_{i+1} &\mapsto [x, y]H_{i+2} \end{aligned}$$

is a $\mathbb{Z}_p\langle\alpha\rangle$ -module homomorphism, and its image is open in A_{i+1} (by definition of the isolated lower central series). Write $\zeta_1 = \zeta$, and proceed by induction on i : suppose that $\zeta_i = \zeta^i$. Now, for any positive integers a and b , we have

$$[x^a, y^b]H_{i+2} = [x, y]^{ab}H_{i+2}$$

by [4, 0.2(i), (ii)] and by using the fact that $[x, y]H_{i+2}$ is central in H/H_{i+2} . Hence, by continuity, this is true for any $a, b \in \mathbb{Z}_p$, and so

$$\begin{aligned} \alpha([x, y]H_{i+2}) &= [x^\zeta, y^{\zeta^i}]H_{i+2} \\ &= [x, y]^{\zeta^{i+1}}H_{i+2}. \end{aligned} \quad \square$$

We deduce:

Corollary 4.4. Let G be a nilpotent-by-finite, orbitally sound compact p -adic analytic group, and H an open normal subgroup of G containing Δ^+ such that H/Δ^+ is nilpotent p -valuable. Then the conjugation action of G on H induces an action of G on H/H_2 given by the map $\xi_1 : G \rightarrow t(\mathbb{Z}_p^\times) \leq \text{Aut}(H/H_2)$ defined above. Moreover, $H \leq \ker \xi_1$. \square

Remark. If $N = H/\Delta^+$ is p -saturable, we may take its corresponding Lie algebra L by Lazard's isomorphism of categories [5, IV, 3.2.6]. As in [2, proof of lemma 8.5]: using [2, lemma 4.2] and the fact that the N/N_i are torsion-free, we can pick an *ordered basis* [5, III, 2.2.4] B for N which is *filtered* relative to the filtration on N : that is,

$$B = \{n_1, n_2, \dots, n_e\},$$

and there exists a filtration of sets

$$B = B_1 \supset B_2 \supset \dots \supset B_{r-1} \neq \emptyset$$

such that B_i is an ordered basis for N_i for each $1 \leq i \leq r-1$. We may order the elements so that, for some integers $1 = k_1 < k_2 < \dots < k_{r-1} < e$, we have $B_i = \{n_{k_i+1}, \dots, n_e\}$ for each $1 \leq i \leq r-1$. Taking logarithms of these basis elements gives us a basis for L , and then Lemma 4.3 implies that, with respect to this basis, the automorphism of L induced by α has the special block lower triangular form

$$\begin{pmatrix} \zeta I_{d_1} & 0 & 0 & \dots & 0 \\ * & \zeta^2 I_{d_2} & 0 & \dots & 0 \\ * & * & \zeta^3 I_{d_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & \zeta^{r-1} I_{d_{r-1}} \end{pmatrix},$$

where $d_i = \text{rk}(L_i/L_{i+1}) = \text{rk}(M_i)$ and I denotes the identity matrix.

5 The finite-by-(nilpotent p -valuable) radical

Let G be a nilpotent-by-finite compact p -adic analytic group. Consider the set

$$\mathcal{S}(G) = \left\{ H \trianglelefteq_O G \mid H/\Delta^+(H) \text{ is nilpotent and } p\text{-valuable} \right\},$$

where “ $H \trianglelefteq_O G$ ” means “ H is an open normal subgroup of G ”. $\mathcal{S}(G)$ is nonempty, as we can pick an open normal nilpotent uniform subgroup of G by [4, 4.1], and hence contains a maximal element. We will show that this maximal element is *unique*, and we will call this element the *finite-by-(nilpotent p -valuable) radical* of G , and once we have shown its uniqueness we will denote it by $\mathbf{FN}_p(G)$.

Remark. Once we have shown the existence and uniqueness of $\mathbf{FN}_p(G)$, it will be clear that it is a *characteristic* open subgroup of G (as automorphisms of G leave $\mathcal{S}(G)$ invariant), and contained in $\text{nio}(G)$ (by Corollary 2.4 and Theorem A).

The quotient group

$$\text{nio}(G)/\mathbf{FN}_p(G)$$

is isomorphic to a subgroup of $t(\mathbb{Z}_p^\times)$ by Corollary 4.4. When $p > 2$, $t(\mathbb{Z}_p^\times)$ is a p' -group, and so $\mathbf{FN}_p(G)/\Delta^+$ is the unique Sylow pro- p subgroup of $\text{nio}(G)/\Delta^+$. (This fails for $p = 2$: the “2-adic dihedral group” $G = \mathbb{Z}_2 \rtimes C_2$ has $\Delta^+(G) = 1$, $\text{nio}(G) = G$, and is its own Sylow 2-subgroup, but $\mathbf{FN}_p(G) = \mathbb{Z}_2$.)

In looking for maximal elements H of $\mathcal{S}(G)$, we may make an immediate simplification. By maximality, any such H must have $\Delta^+(H) = \Delta^+$, i.e. maximal elements of $\mathcal{S}(G)$ are in one-to-one correspondence with maximal elements of

$$\mathcal{S}'(G) = \left\{ H \trianglelefteq_O G \mid \Delta^+ \leq H, H/\Delta^+ \text{ is nilpotent and } p\text{-valuable} \right\},$$

and this set is clearly in order-preserving one-to-one correspondence with the set $\mathcal{S}(G/\Delta^+)$. Hence we may immediately assume without loss of generality that $\Delta^+ = 1$.

Lemma 5.1. Let G be a nilpotent-by-finite compact p -adic analytic group with $\Delta^+ = 1$. Then

- (i) there exists a nilpotent uniform open normal subgroup H of G which contains Δ ,
- (ii) any such H satisfies the property that $Z(H) = \Delta$.

Proof. First, suppose we are given a nilpotent uniform open normal subgroup H . Take $x \in \Delta$: $\mathbf{C}_G(x)$ is open in G by definition, and so

$$\mathbf{C}_H(x) = \mathbf{C}_G(x) \cap H$$

is open in H . Therefore, for any $h \in H$, we can find some integer k such that $h^{p^k} \in \mathbf{C}_H(x)$. This means that $(x^{-1}hx)^{p^k} = h^{p^k}$, and so by [5, III, 2.1.4], we may take (p^k) th roots inside H to see that $x^{-1}hx = h$. In other words, $x \in \mathbf{C}_\Delta(H)$.

Now suppose further that H contains Δ . Then $x \in Z(H)$. In fact, as we have $Z(H) \leq \Delta(H)$ by definition and $\Delta(H) \subseteq \Delta$ by Lemma 1.3(ii), we see that Δ is all of the centre of H . This establishes (ii).

To prove (i), let N be an open normal nilpotent uniform subgroup [4, 4.1] of G . Form $H = N\Delta$, again an open normal subgroup of G . The first paragraph above shows that $[N, \Delta] = 1$; we also know from Lemma 1.3(v) that Δ is abelian and N is nilpotent. This forces H to be nilpotent and open in G , and to contain Δ in its centre.

It remains only to show that H is uniform. As H is nilpotent, its set $t(H)$ of torsion elements forms a normal subgroup [12, 5.2.7], and if $t(H)$ is non-trivial then $t(H) \cap Z(H)$ must be non-trivial [12, 5.2.1]; but $Z(H) = \Delta$ is torsion-free by Lemma 1.3(v), so H must be torsion-free. Now it is easy to check that H is powerful as in [4, 3.1], so that H is uniform by [4, 4.5]. \square

Lemma 5.2. Let G be a nilpotent-by-finite compact p -adic analytic group. Then $\mathcal{S}(G)$ is closed under finite joins, and hence contains a *unique* maximal element H , which is characteristic as a subgroup of G .

Proof. First, observe that, for an open normal subgroup K of G , we have $K \in \mathcal{S}(G)$ if and only if $\overline{K} \in \mathcal{S}(\overline{G})$ (where bars denote quotient by Δ^+). So we continue to assume without loss of generality that $\Delta^+ = 1$.

Suppose we are given $K, L \in \mathcal{S}(G)$: then it remains to show that $KL \in \mathcal{S}(G)$. As K and L are open and normal, it is obvious that KL is too; and since K and L are also nilpotent, Fitting's theorem [14, 1B, Proposition 15] implies that KL is nilpotent. But now, again by [12, 5.2.7], $t(KL) = \Delta^+(KL) \leq \Delta^+ = 1$ – that is, KL is torsion-free, and hence p -valuable by Lemma 2.3.

Now let H be a maximal element of $\mathcal{S}(G)$. Assume for contradiction that H does not contain every other element of $\mathcal{S}(G)$ as a subgroup. Then we may pick some $L \in \mathcal{S}(G)$ not contained in H , and form $HL \in \mathcal{S}(G)$; but now $H \leq HL$, a contradiction to the maximality of H . So H must be the *unique* maximal element of $\mathcal{S}(G)$.

As the set $\mathcal{S}(G)$ is invariant under automorphisms of G , this maximal element H is characteristic in G . \square

Definition 5.3. Let G be a nilpotent-by-finite compact p -adic analytic group. Its *finite-by-(nilpotent p -valuable) radical* $\mathbf{FN}_p(G)$ is the open characteristic subgroup defined in Lemma 5.2.

Proof of Theorem C. The uniqueness of $H = \mathbf{FN}_p(G)$ is the content of Lemma 5.2. As H is a finite-by-nilpotent closed subgroup of G , Corollary 2.4 and

Theorem B show that H is contained in $\text{nio}(G)$. The inclusion $\Delta \leq H$ follows from Lemma 5.1(i). Finally, the isomorphism from $\text{nio}(G)/H$ to a subgroup of $t(\mathbb{Z}_p^\times)$ is induced by the map

$$\xi_1 : \text{nio}(G) \rightarrow t(\mathbb{Z}_p^\times)$$

of Corollary 4.4, whose kernel contains H . □

Proof of Theorem D. This now follows from Theorem C together with Lemma 4.3. □

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